

Quantum criticality of quasi one-dimensional topological Anderson insulators

Alexander Altland,¹ Dmitry Bagrets,¹ Lars Fritz,^{1,2} Alex Kamenev,³ and Hanno Schmiedt¹

¹*Institut für Theoretische Physik, Universität zu Köln, Zùlpicher StraÙe 77, 50937 Köln, Germany*

²*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, Netherlands*

³*W. I. Fine Theoretical Physics Institute and School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA*

(Dated: February 10, 2014)

We present an analytic theory of quantum criticality in the quasi one-dimensional topological Anderson insulators of class AIII and BDI. We describe the systems in terms of two parameters (g, χ) representing localization and topological properties, respectively. Surfaces of half-integer valued χ define phase boundaries between distinct topological sectors. Upon increasing system size, the two parameters exhibit flow similar to the celebrated two parameter flow describing the class A quantum Hall insulator. However, unlike the quantum Hall system, an exact analytical description of the entire phase diagram can be given. We check the quantitative validity of our theory by comparison to numerical transfer matrix computations.

Compared to the enormous research activity on clean topological insulators (tI), the effects of translational symmetry breaking disorder have begun to draw attention only recently [1–5]. In low dimensions, $d \leq 2$, the addition of disorder to a topological insulator drives a crossover into a topological variant of an Anderson insulator (tAI). The latter has to be topologically charged for the phases carrying different indices in the clean case cannot simply ‘disappear’ even in the presence of disorder strong enough to fill the band gap. This means that the phase transition points emerging when a control parameter μ characterizing the clean system is varied must turn into lines of phase transitions meandering through a phase plane spanned by μ and the disorder strength w (cf. Fig. 1.) For a number of tIs, the ensuing phase diagrams have been portrayed by numerical methods [1–3], and for one-dimensional topological superconductors phase transition points have been identified by transfer matrix techniques [4, 5].

However, the best studied example of a tAI remains the quantum Hall (QH) insulator. Within the QH context, the crucial role played by disorder with regard to criticality, edge state formation, and other phenomena has been appreciated right after the discovery of the effect [6]. Its influence on the universal physics of the QH effect is described by Pruisken’s theory [7], a field theory governed by two parameters (g, χ) , where g is a measure of longitudinal electric conduction, and χ a topological parameter proportional to the Hall conductance. At the bare level, both g and χ are non-universal functions of system parameters, where $g \gg 1$ signifies a ‘weakly disordered’ system, and half integer values $\chi = n + 1/2$ define the demarcation lines between sectors of different topological index. Increasing the system size, the parameters (g, χ) undergo renormalization group flow either towards tAI states $(0, n)$ with vanishing conductance and integer Hall angle, or towards QH transition points $(g^*, n + 1/2)$ at criticality. Unfortunately, the fixed points are buried deeply in the realm of strong coupling, $g^* = \mathcal{O}(1)$, and

so far evade analytical treatment.

The statement made by the present paper is that a nearly identical scenario repeats itself in quasi-1d disordered topological insulators, viz. the \mathbb{Z} -insulators of symmetry class AIII and BDI. The addition of disorder to quasi-1d insulators of *finite* length L creates intra-gap states, which turn the system into a conductor, $g(L) \neq 0$, thus compromising its topological integrity [8]. Upon increasing the system size localization effects kick in, and this leads to the restoration of a self-averaging topological index which now is *stabilized* by the presence of disorder. We describe this reentrance mechanism in terms of a system size dependent flow of an index function χ , play-

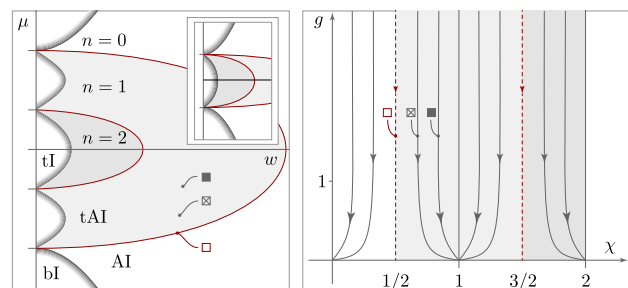


FIG. 1. Left: (μ, w) phase diagram of TI. Hatched areas denote crossover regions between band and Anderson insulator. Red lines mark phase transitions between sectors of different topological index n . The inset describes a situation where the clean system, $w = 0$, has degeneracies, and phase transition points coalesce (e.g. the case of $N > 1$ uncoupled topological chains where inter-chain hopping is solely due to disorder). Right: the corresponding (g, χ) -phase diagram. For bare parameters corresponding to the bulk of a topological phase (cf. the lines marked by a solid and a crossed box), there is exponentially fast flow in the system size L towards an insulating configuration: $g = 0$ and an integer topological index $\chi = n$. At criticality, $\chi + 1/2$ (open box), the flow towards $g = 0$ is algebraic in L , a signature of a critical state.

ing a role analogous to the $2d$ Hall conductance. Our main results are flow diagrams for the two parameters $(g(L), \chi(L))$ strikingly similar to those of the QH system, but unlike these fully tractable by analytic means. We will compare to numerical transfer matrix methods to demonstrate that the flow diagrams accurately describe disorder induced quantum criticality.

Before turning to model specific calculations, let us discuss the structure of the two-parameter flow in qualitative terms. In the absence of disorder, our ‘index’ $\chi = n$ reduces to the standard \mathbb{Z} -valued winding numbers [9] characterizing the clean system. Disorder strong enough to fill the gap, but too weak to localize, $L < \xi$, where $\xi = Nl$ is the localization length, $l \sim |w|^2$ the elastic scattering mean free path, and N the number of quantum channels carried by the system, renders the system effectively metallic with Ohmic conductance $g(L) = \xi/L$. In the metallic regime, the index $\chi(L) = \tilde{\chi}(\mu, w, \dots) \neq n$ assumes a real value where (\dots) denotes non-universal dependence on microscopic system parameters. Upon increasing L the system enters a localization regime, $g(L) \sim \exp(-L/\xi(\tilde{\chi}))$, characterized by an effective length scale $\xi(\tilde{\chi}) \sim |\tilde{\chi} - n - 1/2|^{-\nu}$ where ν is a correlation length exponent. Along with the flow $g(L) \rightarrow 0$ towards a tAI configuration, the index $\chi(L)$ flows from its bare value $\tilde{\chi}$ back to the nearest integer $\chi(L) \xrightarrow{L \rightarrow \infty} [\tilde{\chi}] = n$. The exponentially fast two-parameter flow $(g(L), \chi(L)) \rightarrow (0, n) + \mathcal{O}(e^{-L/\xi(\tilde{\chi})})$ is the quasi-1d analog of the 2d class A QH scaling (cf. Fig. 1). Along with the approach to the insulating state the system stabilizes n zero energy edge states at its end. Transitions between distinct topological sectors are marked by half integer values $\tilde{\chi} = n + 1/2$. At criticality the parameter $\chi(L) = \tilde{\chi} = n + 1/2$ remains stationary and algebraic scaling $g(L) \sim L^{-\alpha}$ of the average conductance signifies the presence of a critical delocalized state.

The analogy to QH extends to the formal level in that the quasi-1d insulators, too, are described by a two parameter field theory. On the bare level, the theory is determined by the pair $(\tilde{\xi}, \tilde{\chi})$ describing the system at length scales $l < L < \xi$. Technically, these constants are computed from a self-consistent Born approximation (SCBA) to the Green function, and criticality is detected by probing half-integerness of $\tilde{\chi}$. We confirm numerically that this procedure accurately describes the phase diagram for given models of disorder. The running observables $(g(L), \chi(L))$ are then extracted by probing the sensitivity of the theory to twists in real space boundary conditions, an operation analogous to Pruisken’s ‘background field’ method [10, 11]. Before turning to the BDI-insulator we discuss the somewhat simpler AIII system.

AIII-insulator — Consider a system of N chains of length L described by the Hamiltonian, $\hat{H} = \sum_l \left[C_l^\dagger ((\mu + t) + (\mu - t)\hat{P})C_{l+1} + C_l^\dagger \hat{V}_l C_{l+1} \right] + \text{h.c.}$, where $C_l = \{c_{l,j}\}$ is a vector of fermion creation opera-

tors, $j = 1, \dots, N$ is the chain index, $l = 1, \dots, L$ labels chain sites, the intra-chain hopping is staggered by the parameter $a = |\mu| - t$ and $\hat{P}c_{l,j} = (-)^l c_{l,j}$ acts as a ‘parity operator’. The matrix \hat{V}_l describes the random inter-chain hopping, which is Gaussian distributed with correlators $\langle V_l^{ij} (V_{l'}^{i'j'})^* \rangle = (w^2/N) \delta_{ij} \delta_{i'j'} \delta_{ll'}$. The anti-commutativity of our (time-reversal non-invariant) Hamiltonian with sublattice parity, $[\hat{H}, \hat{P}]_+ = 0$ indicates that the system belongs to symmetry class AIII. In the absence of disorder, a topological index, or ‘winding number’ may be defined as [9] $n \equiv -i \sum_{q=1}^N \int_0^{2\pi} \frac{dk}{2\pi} \text{tr} (Q^{-1} \partial_k Q)$, where $Q \equiv H_{+-}$ is the block of the Hamiltonian connecting sites of positive and negative parity, and k, q are Fourier indices, $c_{k,q} = \frac{1}{LN} \sum_{l,j} e^{i(kl/L + q2\pi j/N)} c_{l,j}$. Assuming real hopping amplitude $t > 0$ for simplicity, a straightforward calculation shows that $n = N\Theta(t - |\mu|)$, which identifies the amplitude μ as a topological control parameter triggering a transition from $n = 0$ to $n = N$ at $\mu = \pm t$. [12] To access the index n in a way not tied to translational invariance, we imagine the chains compactified to a ring of radius L and pierced by a staggered flux ϕ_0 which affects the fermions as $c_{l,j} \rightarrow e^{i\hat{P}\phi_0 l/L} c_{l,j}$ and the momentum representation of the chiral blocks as $Q(k) \rightarrow Q(k + \phi_0)$ and $Q^\dagger(k) \rightarrow Q^\dagger(k - \phi_0)$, resp. Now consider the generating function $Z(\phi) = \det(\hat{G}_0^{-1}(\phi_0))/\det(\hat{G}_0^{-1}(-i\phi_1))$, where $\phi = (\phi_0, \phi_1)^T$ and $\hat{G}_\epsilon(\phi_\alpha) = (\epsilon^+ - \hat{H}(\phi_\alpha))^{-1}$ is the retarded Green function of the gauged Hamiltonian. The transformation of Q then implies the representation $n \equiv \chi = \frac{i}{2} \partial_{\phi_0} \Big|_{\phi=(0,0)} Z(\phi)$, which no longer relies on the momentum space language. In this formula we also anticipate that in non-translational invariant systems n may generalize to a non-integer parameter χ .

Field theory — We next ask how the system responds to the presence of disorder. The generating function averaged over a Gaussian distribution of the bond amplitudes $V_{j,l}$ affords a representation in terms of a nonlinear σ -model, $Z(\phi) = \int \mathcal{D}T \exp(-S[T])$, with the action [13]

$$S[T] = \int_0^L dx \left[\frac{\tilde{\xi}}{4} \text{str}(\partial_x T \partial_x T^{-1}) + \tilde{\chi} \text{str}(T^{-1} \partial_x T) \right]. \quad (1)$$

Here, ‘str’ is the supertrace and $T = U \begin{pmatrix} e^{y_1} & \\ & e^{iy_0} \end{pmatrix} U^{-1}$ is a 2×2 supermatrix field parameterizing the field space $\text{GL}(1|1)$ [14] in terms of two radial coordinates $y_1 \in \mathbb{R}$, $y_0 \in [0, 2\pi[$ and two Grassmann valued angular variables ρ, σ , where $U = \exp(\sigma^\rho)$. The theory contains two coupling constants, the localization length ξ and the coefficient of the topological term $\tilde{\chi}$. The latter is computed from the underlying microscopic theory as [13] $\tilde{\chi} = \frac{i}{2} \text{tr}(\hat{G}^+ P \partial_k \hat{H})$, where \hat{G}^+ now stands for the zero energy Green function averaged over disorder within self consistent Born approximation. For vanishingly weak disorder, $G^+ \rightarrow -\hat{H}^{-1}$ and $\tilde{\chi} \rightarrow n$ reduces

to the winding number. The value of $\tilde{\chi}$ in the presence of disorder depends on model specifics and will be discussed in more concrete terms below. The external parameters ϕ enter the theory through a twisted boundary condition, $T(L) = \text{diag}(e^{i\phi_1}, e^{i\phi_0})$, and $T(0) = \mathbb{1}$. We finally note that the boundary shift in the non-compact ‘angle’, ϕ_1 enables us to extract the conductance of the wire as [15, 16] $g = (\partial_0^2 + \partial_1^2)|_{\phi=(0,0)}Z$, where $\partial_\alpha = \partial_{\phi_\alpha}$ for brevity. The rationale behind this expression is that the second derivatives $\partial_\alpha^2 Z \sim \partial_\phi^2 \ln \det(G_0(\phi_\alpha))$ probe the average ‘curvature’ of the ϕ -dependent energy levels which, according to Thouless [17], is a measure of the system’s conductance.

Our goal is to understand the scaling of the observables (g, χ) in dependence on the system size L . In the metallic regime $l < L \ll \tilde{\xi}$, ‘size quantization’ of the operator $\partial_x T \partial_x T^{-1}$ suppresses fluctuations of T . Substitution of the minimal configuration compatible with the boundary conditions, $T(x) = \text{diag}(e^{i\phi_1 x/L}, e^{i\phi_0 x/L})$ then yields $(g, \chi) = (\tilde{\xi}/L, \tilde{\chi})$, i.e. an Ohmic conductance, and a topological index set by the bare value $\tilde{\chi}$. To understand what happens upon entering the localization regime, $L \gtrsim \tilde{\xi}$, it is convenient to think of x as imaginary time, and of $Z(\phi)$ as the path integral describing the free motion (first term in the action Eq. (1)) of a fictitious quantum particle on the manifold $\text{GL}(1|1)$ in the presence of a constant gauge flux (the second term). The time-dependent ‘Schrödinger’ equation corresponding to the path integral [13]

$$\tilde{\xi} \partial_x \Psi(y, x) = \frac{1}{J(y)} (\partial_\alpha - iA_\alpha) J(y) (\partial_\alpha - iA_\alpha) \Psi(y, x), \quad (2)$$

is governed by the Laplacian, $J^{-1} \partial_\alpha J \partial_\alpha$ on $\text{GL}(1|1)$, where $J(y) = \sinh^{-2}(\frac{1}{2}(y_1 - iy_0))$ is a Jacobian and the coupling to the vector potential, $A = \tilde{\chi}(1, i)^T$ represents the topological term. From Eq. (2) we obtain the partition function as $Z(\phi) = \Psi(\phi, L)$. The equation can be solved by spectral decomposition $\Psi(\phi, L) = 1 + \sum_{l_0 \in \mathbb{Z}+1/2} \int \frac{d\tilde{\chi}}{(2\pi)^2} P_l \Psi_l(\phi) e^{-\epsilon_l L/\tilde{\xi}}$, where ‘1’ is by supersymmetric normalization of the partition function $Z(0, x) = 1$, and $P_l = 4\pi/(l_0 + il_1)$ implements the spectral decomposition of the initial condition $\Psi(\phi, x \rightarrow 0)$. The eigenfunctions of Eq. (2) are given by $\Psi_l(y) = \sinh(\frac{1}{2}(y_1 - iy_0)) e^{il_\alpha y_\alpha}$, and the corresponding eigenvalues by $\epsilon_l = (l_0 - \tilde{\chi})^2 + (l_1 - i\tilde{\chi})^2$. From this representation, it is straightforward to compute the first and second order expansions in ϕ to arrive at the result

$$g = \sqrt{\frac{\tilde{\xi}}{\pi L}} \sum_{l_0 \in \mathbb{Z}+1/2} e^{-(l_0 - \tilde{\chi})^2 L/\tilde{\xi}}, \quad (3)$$

$$\chi = n - \frac{1}{4} \sum_{l_0 \in \mathbb{Z}+1/2} \left[\text{erf}\left(\sqrt{\frac{L}{\tilde{\xi}}} (l_0 - \delta\tilde{\chi})\right) - (\delta\tilde{\chi} \leftrightarrow -\delta\tilde{\chi}) \right],$$

where $\delta\tilde{\chi} = \tilde{\chi} - n$ is the deviation of $\tilde{\chi}$ off the nearest integer value, n . For generic bare values $(\tilde{\xi}, \tilde{\chi})$ the

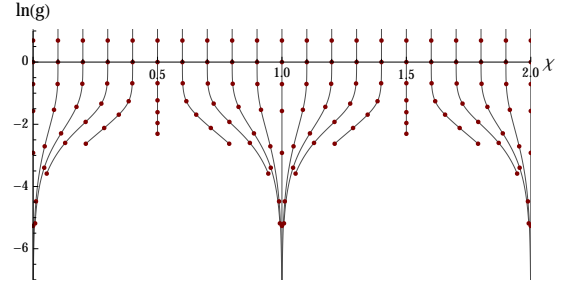


FIG. 2. Flow of the conductance g and the topological parameter χ as a function of system size. Dots are for values, $L/\tilde{\xi} = \frac{1}{4}, \frac{1}{2}, 1, 2, \dots, 32$.

two formulas describe an exponentially fast approach towards an insulating state $(0, n)$ upon increasing length L . At criticality, $(\tilde{\xi}, n + 1/2)$, the topological angle remains invariant, while an algebraic decay of the conductance $g(L) \approx (\tilde{\xi}/\pi L)^{1/2}$ signifies the presence of a delocalized state at the band center. The emergence of power law scaling at criticality can be described in terms of an effective correlation length $\xi(\chi) = \tilde{\xi}|\chi - n - 1/2|^{-\nu}$. Comparing the ansatz, $g \sim \exp(-L/\xi(\chi))$, to the result above, we identify the correlation length exponent $\nu = 2$ [18]. A number of flow lines are shown graphically in Fig. 2, which is the 1d analogue of the two-parameter flow diagram [19] describing criticality in the integer QH system.

Class BDI — We next extend our discussion to the presence of time reversal, symmetry class BDI. System of this type are realized, e.g. [20], as lattice p -wave superconductors with Hamiltonian $\hat{H} = \sum_{l=1}^L [C_l^\dagger \hat{H}_{0,l} C_l + (C_l^\dagger \hat{H}_{1,l} C_{l+1} + \text{h.c.})]$, where the spinless fermion operators $C_l = (c_{l,j}, c_{l,j}^\dagger)^T$ are vectors in channel and Nambu spaces. The on-site part of the Hamiltonian, $\hat{H}_{0,l} = (\mu/2 + \hat{V}_l)\sigma_3$ contains the chemical potential μ and real symmetric inter-chain matrices \hat{V}_l ; σ_i acting in Nambu space. The contribution, $\hat{H}_{1,l} = -\frac{1}{2}t_l\sigma_3 + \frac{i}{2}\hat{\Delta}_l\sigma_2$, contains nearest neighbor hopping, t_l , and the order parameter, $\hat{\Delta}_l$, here assumed to be imaginary for convenience. Quantities carrying a subscript ‘ l ’ may contain site-dependent random contributions. The first quantized representation of \hat{H} obeys the chiral symmetry $[\hat{P}, \hat{H}]_+ = 0$, with $\hat{P} = \sigma_1$. The clean system supports $n \leq N$ Majorana end states, where n decreases upon increasing μ . Generalizing to the presence of disorder, we obtain a pattern of phase transition lines similar to the one discussed above. Before turning to field theory, we apply transfer matrix methods to a numerical description of the ensuing phase portrait.

Transfer Matrix — Defining a doublet $\eta_l = (\psi_{l+1}, \psi_l)^T$, one verifies that the zero energy eigenfunctions, ψ_l of the lattice Hamiltonian obey the recursion relation $\eta_{l+1} = \mathcal{T}_l \eta_l$, where $\mathcal{T}_l = \begin{pmatrix} -H_{1,l}^{-1}H_{0,l} & -H_{1,l}^{-1}H_{1,l}^\dagger \\ 1 & 0 \end{pmatrix}$, and we assumed non-degeneracy of the hopping matrices

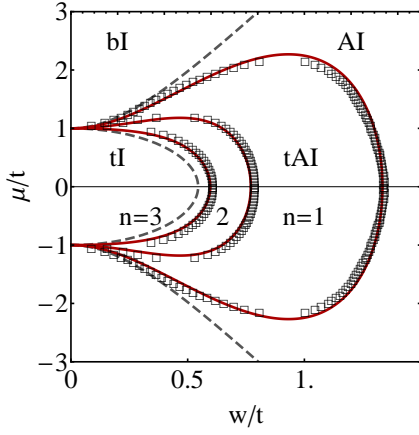


FIG. 3. Phase diagram of the class BDI 3-channel wire. Dashed lines show crossover regions between bI and AI or tI and tAI phases, derived from the SCBA. Solid lines correspond to half integer values of the SCBA computed index $\tilde{\chi}$ and mark boundaries between phases of different n . BI and AI have $n = 0$, while $n > 0$ for tI and tAI. Data points are phase boundaries found from a numerical analysis of Lyapunov exponents λ_j .

$\{H_{1,l}\}$. We iterate this equation to obtain $\eta_L = \mathcal{T}\eta_1$, where $\mathcal{T} = \prod_l^L \mathcal{T}_l$ is the transfer matrix. The presence of a chiral structure means that $\mathcal{T} = \text{bdiag}(\mathcal{T}^{11}, \mathcal{T}^{22})$ can be brought to a block-diagonal form. Representing the positive eigenvalues of \mathcal{T}^{11} as $\exp(L\lambda_j)$, $j = 1, \dots, N$, the index n of individual systems is given by the number of negative ‘Lyapunov exponents’, $\lambda_j < 0$ [21]. We numerically compute the average of these numbers by sampling from a Gaussian distribution of on-site potentials, $\hat{V}_l = \hat{V}_{0,l}\sigma_0 + \hat{V}_{3,l}\sigma_3$, with correlators $\langle V_{a,l}^{ij} V_{a,l}^{i'j'} \rangle = (w^2/N)(\delta_{ii'}\delta_{jj'} - (-)^a\delta_{ij'}\delta_{ji'})$, $a = 0, 3$ and for a model with $t = \Delta$. The results are shown in Fig. 3, where boxes indicate changes of the topological number.

Field Theory — The field theoretical description of the system parallels that of the AIII-insulator. Time reversal invariance means that the fields are now 4×4 matrices spanning the coset space $\text{GL}(2|2)/\text{OSp}(2|2)$ [14], the field action is given by Eq. (1) as before. As in class AIII, the topological coupling constant is given by $\tilde{\chi} = \frac{i}{2}\text{tr}(\hat{G}^+ \hat{P} \partial_k \hat{H})$ where the Green function $\hat{G}^+ = (i0 - \hat{H} - \hat{\Sigma})^{-1}$ contains a self energy $\hat{\Sigma} = -i\Sigma_0\sigma_0 + \Sigma_3\sigma_3$. For random \hat{V}_l (as in the transfer matrix study) the latter is determined by the SCBA equation $-i\Sigma_a = i^k w^2 \text{tr}(\hat{G}_{lj,lj}^+ \sigma_a)$, where the trace is over the Nambu space. This algebraic equation can be solved numerically, and as a result we obtain contour lines of half integer $\tilde{\chi}$ providing an excellent approximation to the numerical transfer matrix data (cf. Fig. 3). In select regions of interest the SCBA equations can be solved analytically. For example, we find that the phase transition points on the $(\mu = 0, w)$ abscissa are located at $t(N/(2n+1))^{1/2}$ [4], where $0 \leq n < N$, or that for disor-

der weaker than these values the degenerate phase transition point $(\mu, w) = (\Delta, 0)$ on the clean system ordinate fans out into N lines $(\mu_n, w) = (\Delta + \frac{2w^2}{\Delta}(3 - \frac{4n+2}{N}), w)$.

The observable pair (g, χ) can be extracted from the field theory as in the AIII-theory. The manifold of T -matrices is now spanned by 8 coordinates, three of which, (y_0, y_1, y_2) , $y_0 \in [0, 2\pi[$, $y_1 \in \mathbb{R}$, $y_2 \in \mathbb{R}^+$ play a role analogous to the radial coordinates of the AIII-manifold. Eq. (2) has to be solved at $y = (\phi_0, \phi_1, 0)$, the Jacobian is given by $J(y) = \sinh(2y_2)/(16 \sinh^2(y_1 - iy_0 + y_2) \sinh^2(y_1 - iy_0 - y_2))$, and the vector potential by $A = 2\tilde{\chi}(1, i, 0)^T$. Although we are not able to solve this equation in generality, we observe that at large values of the variables $y_{1,2}$ the sinh-functions simplify to exponentials. The wave functions in this regime, too, show an exponential profile from which the eigenvalues follow as $\epsilon_l = [(l_0 - 2\tilde{\chi})^2 + (l_1 - 2i\tilde{\chi})^2 + l_2^2 + 1]/4$, with integer l_0 and real $l_{1,2}$. While for general $\tilde{\chi}$ the positive real part of ϵ_l signals localization, we have a zero eigenvalue $\epsilon_{(1,0,0)} = 0$ at $\tilde{\chi} = 1/2$. This signals delocalization, and criticality. The (g, χ) flow diagram therefore has the same structure as in Fig. 1b.

Discussion — We have shown that quantum criticality in the quasi 1d \mathbb{Z} -insulators of class AIII, and BDI is governed by a two parameter flow diagram strikingly similar to that of 2d quantum Hall effect. The flow describes the system-size dependence of two parameters – the average conductance, and a topological signature – whose initial values are functions of the system’s microscopic parameters. This correspondence enables us to describe criticality in disordered topological systems in quantitative agreement with numerical simulations. The two parameter field theories behind this picture describe the universal aspects of both the surface physics and localization in the bulk at length scales exceeding the mean free path. We conjecture that the same scenario of two parameter field theory/flow diagram holds for other tAI: 2d superconducting classes C and D and possibly 3d CI, DIII and AIII Anderson insulators. Exploring to which extent the current framework extends to the \mathbb{Z}_2 topological insulators, requires further research.

Acknowledgment Discussions with P. Brouwer are gratefully acknowledged. Work supported by SFB/TR 12 of the Deutsche Forschungsgemeinschaft. AK was supported by NSF grant DMR1306734. LF acknowledges support from the D-ITP consortium, a program of the Netherlands Organization for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW) as well as from the Deutsche Forschungsgemeinschaft under FR 2627/3-1.

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